Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-006-9039-5

On the Boltzmann Equation for Fermi–Dirac Particles with Very Soft Potentials: Averaging Compactness of Weak Solutions

Xuguang Lu[1]

Received March 31, 2005; accepted July 26, 2005
Published Online: March 21, 2006

The paper considers macroscopic behavior of a Fermi–Dirac particle system. We prove the L^1-compactness of velocity averages of weak solutions of the Boltzmann equation for Fermi–Dirac particles in a periodic box with the collision kernel $b(\cos\theta)|v - v_*|^\gamma$, which corresponds to very soft potentials: $-5 < \gamma \le -3$ with a weak angular cutoff: $\int_0^\pi b(\cos\theta)\sin^3\theta d\theta < \infty$. Our proof for the averaging compactness is based on the entropy inequality, Hausdorff–Young inequality, the L^∞-bounds of the solutions, and a specific property of the value-range of the exponent γ. Once such an averaging compactness is proven, the proof of the existence of weak solutions will be relatively easy.

KEY WORDS: Boltzmann equation; Fermi–Dirac particles; coulomb interaction; weak angular cutoff; averaging compactness.

1. INTRODUCTION

The Boltzmann equation under consideration for one species Fermi–Dirac particles is given by (after normalizing a quantum parameter)

$$\partial_t f + v \cdot \nabla_x f = Q_{\rm B}(f), \quad (t, x, v) \in [0, \infty) \times \mathbf{T}^3 \times \mathbf{R}^3 \tag{1}$$

$$Q_{\rm B}(f) = \int_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \sigma)\Pi_{\rm F}(f) d\sigma dv_* \tag{2}$$

$$\Pi_{\rm F}(f) = f' f'_*(1 - f)(1 - f_*) - f f_*(1 - f')(1 - f'_*) \tag{3}$$

[1] Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China e-mail: xglu@math.tsinghua.edu.cn

with initial and spatial periodic boundary conditions: $f|_{t=0} = f_0$,

$$\mathbf{T}^3 = \prod_{i=1}^{3}[-T_i/2,\ T_i/2], \quad 0 < T_i < \infty, \quad i = 1, 2, 3;$$

f_0, f are periodic in $x = (x_1, x_2, x_3)$ with the period $\mathbf{T} = (T_1, T_2, T_3)$ and f satisfies the L^∞-bounds (due to the Pauli's exclusion principle):

$$0 \le f(t, x, v) \le 1, \quad (t, x, v) \in [0, \infty) \times \mathbf{R}^3 \times \mathbf{R}^3. \tag{4}$$

Physical background and derivation of such quantum Boltzmann models can be found in Chap. 17 of Chapman and Cowling,(15) Nordheim,(25) and Uehling and Uhlenbeck.(27) In Eqs. (1)–(3) the solution f is a density of the number of particles at time $t \in [0, \infty)$, position $x \in \mathbf{T}^3$ with velocity $v \in \mathbf{R}^3$, and f_*, f', f'_* stand for the same function f with different velocity variables v_*, v', v'_* respectively:

$$f_* = f(t, x, v_*), \quad f' = f(t, x, v'), \quad f'_* = f(t, x, v'_*)$$

where v, v_*, and v', v'_* are velocities of two particles before and after their collision which conserves the momentum and kinetic energy:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma, \quad \sigma \in \mathbf{S}^2. \tag{5}$$

For a spherical symmetric two-body interaction potential, the (quantum) collision kernel $B(v - v_*, \sigma)$ (or the scattering cross section $B(v - v_*, \sigma)/|v - v_*|$) is a *non-negative* Borel function of $|v - v_*|$ and $\cos\theta \equiv \mathbf{n} \cdot \sigma$ only(6,13–15):

$$B(v - v_*, \sigma) := B(|v - v_*|, \cos\theta), \quad \sigma \in \mathbf{S}^2, \quad \mathbf{n} = (v - v_*)/|v - v_*|.$$

Under integrable or certain locally integrable assumptions on B, the global existence of Eqs. (1)–(3) in the mild or distributional sense for the whole space domain $\mathbf{R}^3_x$ have been proven by Dolbeault(18) and Lions(21) (and their results also hold true for the spatial periodic box $\mathbf{T}^3$). These results were then extended by Alexandre(2) to weak solutions to the case where the kernel B takes the following form (with our notation)

$$B(v - v_*, \sigma) = \text{const.}|v - v_*|^\gamma \frac{(\cos(\theta/2))^{(3+\gamma)/2}}{(\sin(\theta/2))^{(5-\gamma)/2}}, \qquad -3 < \gamma < 1$$

which is a modification of a classical physical model: the inverse power law of molecular interaction potentials (see e.g. Cercignani(13)). The equality sign "=" in the above expression of B can be relaxed to the inequality sign "$\le$" and in this sense the modified part is only the factor $(\cos(\theta/2))^{(3+\gamma)/2}$ which (thanks to the condition $\gamma > -3$) comes from the use of Carleman's representation of the Boltzmann collision integrals(2, 11, 30) (see also Ref. 29, Chap. 2]. For the spatially homogeneous solutions (i.e. solutions that are independent of x), some basic results on equilibrium states and long-time behavior of solutions have been also obtained in (for instance) Refs. 22 and 23 under the usual assumptions of locally integrable cutoff on B.

In this paper we are mainly concerned with the nonlocally integrable kernel: We assume that B satisfy (recall that B is non-negative)

$$B(v - v_*, \sigma) \le |v - v_*|^\gamma \, b(\cos\theta), \qquad -5 < \gamma \le -3 \tag{6}$$

with a weak angular cutoff:

$$A_0 := 2\pi \int_0^\pi b(\cos\theta)\sin^3\theta \, d\theta < \infty. \tag{7}$$

In physics, an interesting case is the Coulomb interaction model for which the collision kernel B without cutoff is given by $\gamma = -3$ and the Rutherford's formula (see Chap. 21 of Bohm[7] and Chap. 1 of Villani[28,29]):

$$B(v - v_*, \sigma) = \text{const.}|v - v_*|^{-3} b(\cos\theta), \quad b(\cos\theta) = \frac{1}{\sin^4(\theta/2)} \tag{8}$$

which, as pointed out by Bohm (Ref. 7, p. 579), "has the unique property that the exact classical theory, the exact quantum theory, and the Born approximation in the quantum theory all yield the same scattering cross sections." Note that the angular cutoff condition (7) does not cover yet the Coulomb model (8). In fact it is a hard problem to establish a weak form of Eqs. (1)–(3) for (8) without angular cutoff. But the cutoff condition (7) preserves the "main part" of the angular singularity in (8) in the sense that if we choose for instance $b(\cos\theta) = \sin^{-4}(\theta/2)\, 1_{\{\varepsilon \le \theta \le \pi\}}$, then $\int_0^\pi b(\cos\theta)\sin^3\theta d\theta \le 16\log(\pi/\varepsilon)$ which changes so slowly with $0 < \varepsilon \ll 1$ that in practice this truncation is not very sensitive to the actual value of ε (see Ref. 7, p. 521).

1.1. About Weak Solutions

To solve Eqs. (1)–(3) for very soft potentials (i.e. $\gamma \le -3$) with or without angular cutoff, certain weak forms of the equation are necessary as shown in the study of classical Boltzmann equation (see e.g. Refs. 5, 20, 28). As usual, one starts by solving Eqs. (1)–(3) in the mild form (under certain cutoff on B):

$$f^\sharp(t, x, v) = f_0(x, v) + \int_0^t Q_{\rm B}(f)^\sharp(\tau, x, v) d\tau \tag{9}$$

where for any function g, $g^\sharp(t, x, v) = g(t, x + tv, v)$. Then applying (9) and the following identity for x-periodic function g

$$\int_{\mathbf{T}^3} g^\sharp(t, x, v)dx = \int_{\mathbf{T}^3} g(t, x, v)dx \qquad \forall\,(t, v) \in [0, \infty) \times \mathbf{R}^3$$

the Eqs. (1)–(3) can be written as the following weak form

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\varphi(t,x,v)dvdx = \int_{\mathbf{T}^3\times\mathbf{R}^3} f_0(x,v)\varphi(0,x,v)dvdx$$
$$+\int_0^t d\tau \int_{\mathbf{T}^3\times\mathbf{R}^3} f(\partial_\tau\varphi + v\cdot\nabla_x\varphi)dvdx$$
$$+\frac{1}{4}\int_0^t d\tau\int_{\mathbf{T}^3\times\mathbf{R}^3} Q_B(f\mid\Delta\varphi)dvdx \quad \forall\, t\geq 0 \tag{10}$$

where $\varphi(t,x,v)$ are smooth functions periodic in x with period $\mathbf{T}$, $\Delta\varphi$ denotes the collisional difference of velocity functions $v\mapsto\varphi(t,x,v)$:

$$\Delta\varphi = \Delta\varphi(v,v_*,v',v'_*) := \varphi(v)+\varphi(v_*)-\varphi(v')-\varphi(v'_*), \tag{11}$$

and

$$Q_B(f\mid\Delta\varphi)(t,x,v) = \int_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\sigma)\Delta\varphi\,\Pi_{\mathrm{F}}(f)d\sigma dv_*. \tag{12}$$

The problem of rigorous validity of the weak form (10) is focused on finding a suitable cutoff condition (as weak as possible) on B such that the collision integral (12) is finite for a large class of smooth functions φ. In the case of classical Boltzmann equation, the angular cutoff condition (7) for the Coulomb model (8) or generally for the very soft potential model (6) was already used in Villani[(28)] (see also Ref. 29, Chap. 2) where the global existence of weak solutions was proven by using the entropy control (the Boltzmann H-theorem). Such weak solutions are then also called H-solutions[(2,28)]. For the present Boltzmann–Fermi–Dirac model (1)–(3), this consideration on entropy control is more important and seems the only choice because the "mixing" effect, i.e. the prime ($'$) in $f'f'_*$ and $(1-f')(1-f'_*)$, cannot disappear simultaneously with any change of velocity variables and thus the following classical relation (e.g. for the original Boltzmann collision integral)

$$\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\sigma)(f'f'_*-ff_*)\Delta\varphi\,d\sigma dvdv_*$$
$$= -2\int_{\mathbf{R}^3\times\mathbf{R}^3} ff_*\left(\int_{\mathbf{S}^2} B(v-v_*,\sigma)\Delta\varphi\,d\sigma\right)dvdv_*,$$

which preserves the "main part" of the angular singularity as shown above for (8), could not apply to the Fermi–Dirac model. While the entropy method still works; it combines with the L^∞-bounds (4) and the "very softness" $-5<\gamma\leq-3$ enables us also to avoid dealing with the problem of the boundedness of the integrals like

$$\int_{\mathbf{T}^3}\left(\int_{\mathbf{R}^3} f(t,x,v)(1+|v|^2)dv\right)^2 dx \ \text{ or } \int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)|v|^{2+\delta}dvdx, \quad \delta>0.$$

(See Lemma 2 in Section 2.)

Before introducing our weak solutions of the Eqs. (1)–(3), let us first show the role of entropy control in dealing with the singularity of collision integrals. Recall that the entropy functional $S(f)$ for the Boltzmann–Fermi–Dirac model is given by

$$S(f)=\int_{\mathbf{T}^3\times\mathbf{R}^3}(-(1-f)\log(1-f)-f\log f)\,dvdx$$

for measurable functions f satisfying (4) and $f\in L^1_2(\mathbf{T}^3\times\mathbf{R}^3)$, where

$$L^1_s(\mathbf{T}^3\times\mathbf{R}^3)=\left\{f\ \middle|\ \|f\|_{L^1_s}:=\int_{\mathbf{T}^3\times\mathbf{R}^3}|f(x,v)|(1+|v|^2)^{s/2}dvdx<\infty\right\}.$$

Since $0\le f\le 1$ implies that

$$|(1-f)\log(1-f)|+|f\log f|\le f(1+|v|^2)+e^{-|v|^2/2}$$

the entropy $S(f)$ is bounded:

$$0\le S(f)\le \|f\|_{L^1_2}+C_0|\mathbf{T}^3| \tag{13}$$

where $|\mathbf{T}^3|=T_1T_2T_3$ denotes the volume of the box $\mathbf{T}^3$. Classical derivation shows at least formally that a solution $f(t)=f(t,\cdot,\cdot)$ of Eqs. (1)–(3) with initial datum $f|_{t=0}=f_0$ satisfies the entropy identity

$$S(f(t))=S(f_0)+\int_0^t d\tau\int_{\mathbf{T}^3}D(f)(\tau,x)dx,\quad t\ge 0. \tag{14}$$

In the present paper we shall use only the entropy inequality

$$S(f(t))\ge S(f_0)+\int_0^t d\tau\int_{\mathbf{T}^3}D(f)(\tau,x)dx,\quad t\ge 0. \tag{15}$$

Here

$$D(f)=\frac{1}{4}\int_{\mathbf{R}^6\times\mathbf{S}^2}B(v-v_*,\sigma)\Gamma(f)d\sigma dvdv_* \qquad \text{(entropy dissipation)},$$

$$\Gamma(f)=(f'f'_*(1-f)(1-f_*)-ff_*(1-f')(1-f'_*))\log\left(\frac{f'f'_*(1-f)(1-f_*)}{ff_*(1-f')(1-f'_*)}\right),$$

and we define

$$(a-b)\log\left(\frac{a}{b}\right)=+\infty \quad \text{for } a>0=b \text{ or } a=0<b;\quad =0 \quad \text{for } a=b=0.$$

To establish weak solutions we use the following test function space $C^1_{b,\mathbf{T}}([0,\infty)\times\mathbf{R}^3\times\mathbf{R}^3)$ defined by

$$\varphi \in C^1_{b,\mathbf{T}}([0,\infty)\times\mathbf{R}^3\times\mathbf{R}^3) \Longleftrightarrow \varphi\in C^1([0,\infty)\times\mathbf{R}^3\times\mathbf{R}^3),$$

$$\sup_{(t,x,v)\in[0,\infty)\times\mathbf{T}^3\times\mathbf{R}^3} (|\varphi|+|\partial_t\varphi|+|\nabla_x\varphi|+|\nabla_v\varphi|)(t,x,v)<\infty,$$

and the function $x\mapsto\varphi(t,x,v)$ is periodic with period $\mathbf{T}=(T_1,T_2,T_3)$.

Now suppose that f is a solution of Eqs. (1)–(3) (in the mild sense (9) for instance) satisfying the entropy inequality (15) and conserving the mass, momentum and energy. Then

$$\|f(t)\|_{L^1_2}=\|f_0\|_{L^1_2}<\infty, \qquad S(f(t))\le\|f_0\|_{L^1_2}+C_0|\mathbf{T}^3|, \qquad t\ge 0.$$

Let $\varphi\in C^1_{b,\mathbf{T}}([0,\infty)\times\mathbf{R}^3\times\mathbf{R}^3)$, $0\le t_1<t_2<\infty$. By elementary inequality $|\Delta\varphi|\le\sqrt{2}\,\|\nabla\varphi\||v-v_*|\sin\theta$ (see Lemma 1 in Section 2) we have

$$\int_{t_1}^{t_2}dt\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2}B\,|\Delta\varphi|\,|\Pi_{\mathrm{F}}(f)|d\sigma dv_*dvdx$$

$$\le\sqrt{2}\,\|\nabla\varphi\|\int_{t_1}^{t_2}dt\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2}B\,|v-v_*|\sin\theta\,|\Pi_{\mathrm{F}}(f)|d\sigma dv_*dvdx. \tag{16}$$

We show that

$$\int_{t_1}^{t_2}dt\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2}B(v-v_*,\sigma)|v-v_*|\sin\theta|\Pi_{\mathrm{F}}(f)|d\sigma dv_*dvdx$$

$$\le\left(\frac{C_0A_0\,(t_2-t_1)}{5-|\gamma|}\|f_0\|_{L^1_2}\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbf{T}^3}D(f)(t,x)dxdt\right)^{1/2}. \tag{17}$$

In particular (using the entropy inequality (15)) for all $0<T<\infty$

$$\int_0^T dt\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2}B(v-v_*,\sigma)|v-v_*|\sin\theta|\Pi_{\mathrm{F}}(f)|d\sigma dv_*dvdx$$

$$\le\left(\frac{C_0A_0\,T}{5-|\gamma|}\|f_0\|_{L^1_2}\right)^{1/2}[S(f(T))-S(f_0)]^{1/2}. \tag{18}$$

Here and below C_0 always denotes positive absolute constants.

To do this we consider the decomposition

$$\Pi_{\mathrm{F}}(f)=\Pi_1(f)\Pi_2(f)$$

where

$$\Pi_1(f)=\sqrt{f'f'_*(1-f)(1-f_*)}+\sqrt{ff_*(1-f')(1-f'_*)},$$
$$\Pi_2(f)=\sqrt{f'f'_*(1-f)(1-f_*)}-\sqrt{ff_*(1-f')(1-f'_*)}.$$

By Cauchy–Schwarz inequality we have

$$\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|v-v_*|\sin\theta\ |\Pi_{\mathrm{F}}(f)|d\sigma dv_* dv dx$$
$$\le \left(\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|v-v_*|^2\sin^2\theta\,|\Pi_1(f)|^2 d\sigma dv_* dv dx\right)^{1/2}$$
$$\times\left(\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\Pi_2(f)|^2 d\sigma dv_* dv dx\right)^{1/2}.$$

For the first factor in the right hand side of this inequality we use the following estimate (see Lemma 2 below)

$$\int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{f(t,x,v)f(t,x,v_*)}{|v-v_*|^{|\gamma|-2}} dv dv_* \le \frac{C_0}{5-|\gamma|}\int_{\mathbf{R}^3} f(t,x,v)|v|^{5-|\gamma|}dv$$

with the condition $0 < 5-|\gamma| \le 2$ and note that $|\Pi_1(f)|^2 \le 2(f'f'_* + ff_*)$ to obtain

$$\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|v-v_*|^2\sin^2\theta\,|\Pi_1(f)|^2 d\sigma dv_* dv dx$$
$$\le 4\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|v-v_*|^2\sin^2\theta f f_* d\sigma dv_* dv dx$$
$$\le 4A_0\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{R}^3} \frac{ff_*}{|v-v_*|^{|\gamma|-2}} dv_* dv dx \le \frac{C_0A_0(t_2-t_1)}{5-|\gamma|}\|f_0\|_{L^1_2}. \qquad (19)$$

For the second factor we use the elementary inequality[28]

$$(\sqrt{a}-\sqrt{b})^2 \le \frac{1}{4}(a-b)\log(\frac{a}{b}), \qquad 0\le a,b<\infty$$

to deduce $|\Pi_2(f)|^2 \le \frac{1}{4}\Gamma(f)$ and thus

$$\int_{t_1}^{t_2} dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\Pi_2(f)|^2 d\sigma dv_* dv dx \le \int_{t_1}^{t_2} dt \int_{\mathbf{T}^3} D(f)(t,x)dx. \qquad (20)$$

This gives (17).

Inequality (17) (therefore (18)) holds rigorously at leat for mild solutions f^n of Eqs. (1)–(3) with strong cutoff kernels B_n. And one can construct B_n such that $B_n \nearrow B\,(n\to\infty)$ and f^n satisfy the conservation of mass, momentum and energy and satisfy the entropy identity (14) with the entropy dissipation $D_n(f^n)$ corresponding to B_n. By taking weak limit, the inequality (18) holds also for a limiting function f which is expected to be a weak solution of Eqs. (1)–(3) in the following sense:

1.2. Definition of Weak Solutions of Eqs. (1)–(3)

Let B satisfy (6) and (7). Let f_0 be measurable and x-periodic function (with the period $\mathbf{T}$) satisfying $0 \le f_0 \le 1$ on $\mathbf{R}^3 \times \mathbf{R}^3$ and $f_0 \in L^1_2(\mathbf{T}^3 \times \mathbf{R}^3)$. Let f be a measurable and x-periodic function (with the period $\mathbf{T}$) on $[0, \infty) \times \mathbf{R}^3 \times \mathbf{R}^3$ and satisfy the L^∞-bounds (4). We say that f is a weak solution to the Eqs. (1)–(3) with the initial datum f_0 if f satisfies the following (i)–(ii):

(i) For any $t \ge 0$, $(x, v) \mapsto f(t, x, v)$ is measurable on $\mathbf{R}^3 \times \mathbf{R}^3$, $f|_{t=0} = f_0$, and $\sup_{t\ge 0} \|f(t)\|_{L^1_2} < \infty$.

(ii) For any $\varphi \in C^1_{b,\mathbf{T}}([0, \infty) \times \mathbf{R}^3 \times \mathbf{R}^3)$ and for any $0 < T < \infty$,

$$\int_0^T dt \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B(v - v_*, \sigma)|\Delta\varphi||\Pi_{\mathrm{F}}(f)|d\sigma dv_* dv dx < \infty \tag{21}$$

and (f, φ) satisfies the equation (10).

Note that the integrability condition (21) can be obtained by the inequalities (16) and (18) provided that f satisfies (18).

1.3. Conservation of Energy

It is worth noting that the very soft potentials ($-5 < \gamma \le -3$) possess an advantage that if the angular function $b(\cdot)$ satisfies a stronger cutoff condition

$$A_0^* := 2\pi \int_0^\pi b(\cos\theta)\sin^2\theta d\theta < \infty$$

then every weak solution to Eqs. (1)–(3) conserves the energy. (The conservation of the mass is obvious and the conservation of the momentum follows from the following argument.) To show this we choose a family $\{\varphi_\varepsilon\}$ of test functions

$$\varphi_\varepsilon(v) = \frac{1}{\varepsilon}(1 - e^{-\varepsilon|v-v_0|^2}), \quad \varepsilon > 0, \quad v_0 \in \mathbf{R}^3$$

and compute $\partial^2_{v_i v_j}\varphi_\varepsilon(v) = -4\varepsilon e^{-\varepsilon|v-v_0|^2}(v - v_0)_i(v - v_0)_j + 2e^{-\varepsilon|v-v_0|^2}\delta_{ij}$ which implies $|\partial^2_{v_i v_j}\varphi_\varepsilon(v)| \le 4e^{-\varepsilon|v-v_0|^2}(1 + \varepsilon|v - v_0|^2) \le 4$ and so by Lemma 1 (see Section 2) we have

$$|\Delta\varphi_\varepsilon| \le 6|v - v_*|^2 \sin\theta \quad \forall\, \varepsilon > 0.$$

Then applying inequality $|\Pi_{\mathrm{F}}(f)| \le f' f'_* + f f_*$ with the same argument as in

(19) we obtain the integrability:

$$\begin{aligned}&\int_0^t d\tau \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|v-v_*|^2 \sin\theta |\Pi_{\mathrm{F}}(f)| d\sigma dv_* dv dx \\ &\quad \le 2\int_0^t d\tau \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|v-v_*|^2 \sin\theta f f_* d\sigma dv_* dv dx \\ &\quad \le \frac{C_0 A_0^* t}{5-|\gamma|} \sup_{\tau\in[0,t]} \|f(\tau)\|_{L^1_2} < \infty \qquad \forall\, 0<t<\infty.\end{aligned}$$

On the other hand we have $0 \le \varphi_\varepsilon(v) \le |v-v_0|^2$, $\lim_{\varepsilon\to 0+} \varphi_\varepsilon(v) = |v-v_0|^2$, and thus $\lim_{\varepsilon\to 0+} |\Delta\varphi_\varepsilon| = 0$. Therefore it follows from dominated convergence theorem and the weak form (10) that for all $0 \le t < \infty$

$$\lim_{\varepsilon\to 0+} \int_0^t d\tau \int_{\mathbf{T}^3\times\mathbf{R}^3} Q_B(f \mid \Delta\varphi_\varepsilon) dv dx = 0,$$

$$\int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)|v-v_0|^2 dv dx = \int_{\mathbf{T}^3\times\mathbf{R}^3} f_0(x,v)|v-v_0|^2 dv dx.$$

This proves the conservation of energy by choosing $v_0 = 0$. Also, since v_0 is arbitrary, the conservation of momentum follows.

1.4. Velocity-Averaging Compactness

The main purpose of this paper is to prove the L^1-compactness of velocity averages (for any given $\Psi \in L^\infty([0,\infty)\times\mathbf{T}^3\times\mathbf{R}^3)$)

$$\langle f\Psi\rangle(t,x) = \int_{\mathbf{R}^3} f(t,x,v)\Psi(t,x,v) dv \tag{22}$$

of weak or approximate solutions f of Eqs. (1)–(3). As is well-known, the compactness of velocity averages is one of several basic tools in dealing with various convergence or stability of solutions or approximate solutions of kinetic equations (see e.g. Refs. 8–10, 16, 17, 19, 29); see also a recent application of the averaging compactness in Ref. 12 where an appendix may be very helpful to the reader), and it is different from some other tools (e.g. estimates of entropy dissipation, strong compactness of the Boltzmann collision gain operators, cancellation lemma, etc., see the review in Ref. 29) that the velocity-averaging compactness depends completely on the related kinetic equation. For the present Boltzmann-Fermi–Dirac model, since for very soft potentials ($-5 < \gamma \le -3$) this dependence on the equation (1) is only through the weak form (10) and the angular cutoff condition (7) on the kernel B is very weak, the known methods for proving the averaging compactness of classical, mild or renormalized solutions of kinetic equations have to be improved for the present type of weak solutions. Once such an averaging compactness is established, the proof of the existence of weak solutions of Eqs. (1)–(3) will be relatively easy.[24]

1.5. About Full Compactness

Although our main result is only concerned with averaging compactness, a very important question is to ask about full compactness (in the whole variables (t, x, v) or at least in (x, v)) of weak or approximate solutions. Once the averaging compactness is established, the tools used in Refs. 3 and 4 (see also Ref. 2) maybe adapt to the Boltzmann–Fermi–Dirac model for very soft potentials . For instance, if in addition to (6)–(7) we assume that B also satisfies

$$B(v - v_*, \sigma) \geq B_0(v - v_*, \sigma) := \left(1 + |v - v_*|^{|\gamma|}\right)^{-1} b(\cos\theta) 1_{\{0 \leq \theta \leq \pi/2\}}$$

and let $D(f)$, $D_0(f)$ be defined above corresponding to B and B_0 respectively, then $D(f) \geq D_0(f)$ and the cancellation argument in Ref. 3 (or the equality (2.2) in Section 2 below) can be used to B_0 to obtain the following inequality: There is an absolute constant $C_0 > 0$ such that for all $f \in L^1(\mathbf{R}^3)$ with $0 \leq f(v) \leq 1$,

$$\int_{\mathbf{R}^6 \times \mathbf{S}^2} B_0 f(f'_* - f_*) d\sigma dv_* dv \leq C_0 A_0 \int_{\mathbf{R}^3} f(v) dv.$$

This inequality holds at least when $b(t)$ is integrable. Then following the proof of Theorem 1 in Ref. 3 (using a monotone approximation to $b(t)$ ...) we can prove that (for instance)

$$\frac{1}{4} \int_{\mathbf{R}^6 \times \mathbf{S}^2} B_0 f(1 - f)(f_* - f'_*)^2 d\sigma dv_* dv \leq D_0(f) + C_0 A_0 \int_{\mathbf{R}^3} f(v) dv, \quad (23)$$

$$\frac{1}{4} \int_{\mathbf{R}^6 \times \mathbf{S}^2} B_0 |f - f'||f_* - f'_*| d\sigma dv_* dv \leq D_0(f) + C_0 A_0 \int_{\mathbf{R}^3} f(v) dv. \quad (24)$$

Since $f(1 - f) \geq 0$, the inequality (23) seems the most applicable for proving the full compactness of weak solutions $\{f^n\}$ of Eqs. (1)–(3) because in view of Ref. 3 one needs only to prove that the averages $(t, x) \mapsto \int_{\mathbf{R}^3} f^n(1 - f^n) dv$ have a certain positive pointwise lower bound. But this is not easy (even use the fact that f^n are solutions) because $f \mapsto f(1 - f)$ is concave, not convex. It is also not clear how to deal with the mixing term $|f - f'||f_* - f'_*|$ in the inequality (24). Note that the left hand sides of (23) and (24) are derived from some detailed ones. We do not know whether these detailed versions can provide the required lower bound. . . .

Before ending this section we need to mention some facts about changes of variables in integration. Recall that the velocities v', v'_* can be also represented by the following ω-representation:

$$v' = v - ((v - v_*) \cdot \omega)\omega, \qquad v'_* = v_* + ((v - v_*) \cdot \omega)\omega. \tag{25}$$

Accordingly in many articles the collision kernel is written as

$$\tilde{B}(v - v_*, \omega) := \tilde{B}(|v - v_*|, |\mathbf{n} \cdot \omega|), \quad \omega \in \mathbf{S}^2, \quad \mathbf{n} = (v - v_*)/|v - v_*|.$$

The relation between $B(v - v_*, \sigma)$ and $\tilde{B}(v - v_*, \omega)$ is given by (see e.g. Ref. 29, Chap. 1)

$$\tilde{B}(|v - v_*|, |\mathbf{n} \cdot \omega|) = 2|\mathbf{n} \cdot \omega| B(|v - v_*|, \mathbf{n} \cdot \sigma), \quad |\mathbf{n} \cdot \omega| = \sqrt{\frac{1 - \mathbf{n} \cdot \sigma}{2}}$$

with $\mathbf{n} = (v - v_*)/|v - v_*|$. It is easily shown that for all non-negative measurable function $\Phi(v, v_*)$ on $\mathbf{R}^3 \times \mathbf{R}^3$ we have

$$\int_{\mathbf{S}^2} B(v - v_*, \sigma)\Phi(v', v'_*)d\sigma = \int_{\mathbf{S}^2} \tilde{B}(v - v_*, \omega)\Phi(v', v'_*)d\omega \tag{26}$$

where (v', v'_*) in the right hand side (resp. the left hand side) is given by the ω-representation (25) (resp. σ-representation (5)). Note that the conservation of kinetic energy $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ says that for any fixed $\omega \in \mathbf{S}^2$ the mapping $(v, v_*) \mapsto (v', v'_*)$ (given by the ω- representation (25)) is a *linear* orthogonal transform on $\mathbf{R}^3 \times \mathbf{R}^3$. It is this linear property and the relation (26) that enable one to deduce various identities about collision integrals with σ-representation. In this paper unless otherwise stated we always use the σ-representation (5).

2. SOME LEMMAS

In this section we collect and prove some lemmas for proving our main results which are given in the next section.

Lemma 1. *Let $\Delta\varphi = \Delta\varphi(v, v_*, v', v'_*)$ be defined in (11). If $\varphi \in C^1_b(\mathbf{R}^3)$ then*

$$|\Delta\varphi| \le \sqrt{2}\, \|\nabla\varphi\|\, |v - v_*| \sin\theta, \quad (v, v_*, \sigma) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2.$$

If $\varphi \in C^2_b(\mathbf{R}^3)$, then

$$|\Delta\varphi| \le \frac{1}{2}\|H_\varphi\| |v - v_*|^2 \sin\theta.$$

Here $\|\nabla\varphi\| = \sup_{v \in \mathbf{R}^3} \left(\sum_{i=1}^3 |\partial_{v_i}\varphi(v)|^2\right)^{1/2}$, $\|H_\varphi\| = \sup_{v \in \mathbf{R}^3} \left(\sum_{i,j} |\partial^2_{v_i v_j}\varphi(v)|^2\right)^{1/2}$.

Proof. By identities

$$|v' - v| = |v'_* - v_*| = |v - v_*| \sin(\theta/2),$$
$$|v' - v_*| = |v'_* - v| = |v - v_*| \cos(\theta/2),$$
$$\Delta\varphi = (\varphi - \varphi') + (\varphi_* - \varphi'_*) = (\varphi - \varphi'_*) + (\varphi_* - \varphi')$$

we obtain the first estimate:

$$|\Delta\varphi| \le 2\|\nabla\varphi\||v - v_*| \min\{\sin(\theta/2),\ \cos(\theta/2)\} \le \sqrt{2}\,\|\nabla\varphi\||v - v_*| \sin\theta.$$

For the second estimate we compute (using $v_* - v'_* = v' - v$)

$$\begin{aligned}\Delta\varphi &= [\varphi(v_*) - \varphi(v'_*)] - [\varphi(v') - \varphi(v)] \\ &= \int_0^1 [\nabla\varphi(v'_* + t(v_* - v'_*)) - \nabla\varphi(v + t(v' - v))] \cdot (v' - v)dt \\ &= \int_0^1\int_0^1 (v'_* - v)^T H_\varphi(v + t(v' - v) + \tau(v'_* - v))(v' - v)d\tau dt\end{aligned}$$

where $H_\varphi(v) = (\partial^2_{v_i v_j}\varphi(v))_{3\times 3}$denotes the Hessian matrix of φ. This gives

$$|\Delta\varphi| \le \|H_\varphi\|\,|v'_* - v||v' - v| = \tfrac{1}{2}\|H_\varphi\|\,|v - v_*|^2 \sin\theta.$$ □

Lemma 2. *Let f, g be measurable functions on $\mathbf{R}^3$ satisfying $0 \le f, g \le 1$ on $\mathbf{R}^3$, and let $\alpha < 3$ be a constant. Then*

$$\int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{f(v)g(v_*)}{|v - v_*|^\alpha} dv dv_* \le \frac{2^{5-\alpha}\pi}{3-\alpha}\int_{\mathbf{R}^3} (f(v) + g(v))|v|^{3-\alpha} dv.$$

Proof. Using the inequality$1 \le 1_{\{|v-v_*|\le 2|v|\}} + 1_{\{|v-v_*|\le 2|v_*|\}}$ we compute

$$\begin{aligned}\int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{f(v)g(v_*)}{|v - v_*|^\alpha} dv dv_* &\le \int_{\mathbf{R}^3} f(v)\left(\int_{\mathbf{R}^3} \frac{1}{|v - v_*|^\alpha} 1_{\{|v-v_*|\le 2|v|\}} dv_*\right) dv \\ &+ \int_{\mathbf{R}^3} g(v_*)\left(\int_{\mathbf{R}^3} \frac{1}{|v - v_*|^\alpha} 1_{\{|v-v_*|\le 2|v_*|\}} dv\right) dv_* \\ &= \frac{2^{5-\alpha}\pi}{3-\alpha}\int_{\mathbf{R}^3} (f(v) + g(v))|v|^{3-\alpha} dv.\end{aligned}$$ □

Lemma 3. *Let $b(t)$, $\Phi(r)$ be non-negative Borel functions on $t \in [-1, 1]$ and $r \in [0, \infty)$ respectively, let $f(v)$ be a non-negative measurable function on $\mathbf{R}^3$.*

Then with $\mathbf{n} = (v - v_*)/|v - v_*|$ *we have, for all* $v \in \mathbf{R}^3$,

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} b(\mathbf{n}\cdot\sigma)\Phi(|v-v_*|)f(v')d\sigma dv_*$$
$$= 2\pi\int_0^{\pi} \frac{b(\cos\theta)\sin\theta}{\sin^3(\theta/2)}\left\{\int_{\mathbf{R}^3}\Phi\left(\frac{|v-v_*|}{\sin(\theta/2)}\right)f(v_*)dv_*\right\}d\theta, \tag{28}$$

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} b(\mathbf{n}\cdot\sigma)\Phi(|v-v_*|)f(v_*')d\sigma dv_*$$
$$= 2\pi\int_0^{\pi} \frac{b(\cos\theta)\sin\theta}{\cos^3(\theta/2)}\left\{\int_{\mathbf{R}^3}\Phi\left(\frac{|v-v_*|}{\cos(\theta/2)}\right)f(v_*)dv_*\right\}d\theta. \tag{29}$$

Proof. Let $\tilde{b}(\tau)$ be defined on $\tau \in [0,1]$ by

$$\tilde{b}(\cos\theta) = 2\cos\theta\, b(-\cos 2\theta), \quad \theta\in[0,\pi/2]. \tag{30}$$

Then it has been proven in Ref. 22 that (with the ω-representation (25))

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} \tilde{b}(|\mathbf{n}\cdot\omega|)\Phi(|v-v_*|)f(v')d\omega dv_*$$
$$= 4\pi\int_0^{\pi/2} \frac{\sin\theta\,\tilde{b}(\cos\theta)}{\cos^3\theta}\left\{\int_{\mathbf{R}^3}\Phi\left(\frac{|v-v_*|}{\cos\theta}\right)f(v_*)dv_*\right\}d\theta,$$

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} \tilde{b}(|\mathbf{n}\cdot\omega|)\Phi(|v-v_*|)f(v_*')d\omega dv_*$$
$$= 4\pi\int_0^{\pi/2} \frac{\sin\theta\,\tilde{b}(\cos\theta)}{\sin^3\theta}\left\{\int_{\mathbf{R}^3}\Phi\left(\frac{|v-v_*|}{\sin\theta}\right)f(v_*)dv_*\right\}d\theta.$$

Since, by (24),

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} b(\mathbf{n}\cdot\sigma)\Phi(|v-v_*|)(f(v'),f(v_*'))d\sigma dv_*$$
$$= \int_{\mathbf{R}^3\times\mathbf{S}^2} \tilde{b}(|\mathbf{n}\cdot\omega|)\Phi(|v-v_*|)(f(v'),f(v_*'))d\omega dv_*,$$

the equalities (28)–(29) follow from the relation (30). □

Remark. *A detailed proof of Lemma 3 (for (29)) can be also found in Ref. 3 (Cancellation Lemma) under the condition that function* $b(t)$ *is supported on* $[0,1]$, *which is not a restriction when connecting the Boltzmann type equations because as explained in Ref. 3 that in such equations one can replace* $B(v-v_*,\sigma)$ *with* $[B(v-v_*,\sigma)+B(v-v_*,-\sigma)]1_{\{\mathbf{n}\cdot\sigma\geq 0\}}$.

Lemma 4. *Let $B(v-v_*,\sigma)$ satisfy the conditions* (6) – (7) *and let f be measurable function on $\mathbf{R}^3$ satisfying* $0 \le f \le 1$. *Then*

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\sigma)\sin^2\theta\,|v-v_*|^{|\gamma|+2} f(v')f(v'_*)d\sigma dv_* \le 2^{5/2}A_0(1+|v|^2)\int_{\mathbf{R}^3}(1+|v_*|^2)f(v_*)dv_*, \quad v\in\mathbf{R}^3.$$

Proof. By assumption on B we have

$$\int_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\sigma)\sin^2\theta\,|v-v_*|^{|\gamma|+2} f(v')f(v'_*)d\sigma \le \int_{\mathbf{R}^3\times\mathbf{S}^2} b(\cos\theta)\sin^2\theta\,|v-v_*|^2 f(v')f(v'_*)d\sigma dv_*.$$

For dealing with singularity in θ we consider a decomposition

$$b(\cos\theta)\sin^2\theta = b_1(\cos\theta)+b_2(\cos\theta),$$

$$b_1(\cos\theta) = b(\cos\theta)\sin^2\theta\,1_{\{\pi/2<\theta\le\pi\}}, \quad b_2(\cos\theta) = b(\cos\theta)\sin^2\theta\,1_{\{0\le\theta\le\pi/2\}}.$$

Then applying Lemma 3 to the functions $b_1(t)$, $b_2(t)$ and $\Phi(r)=r^2$ we compute with the assumption $0\le f\le 1$ that

$$\begin{aligned}
&\int_{\mathbf{R}^3\times\mathbf{S}^2} b(\cos\theta)\sin^2\theta\,|v-v_*|^2 f(v')f(v'_*)d\sigma dv_* \\
&\le \int_{\mathbf{R}^3\times\mathbf{S}^2} b_1(\cos\theta)\,|v-v_*|^2 f(v')d\sigma dv_* + \int_{\mathbf{R}^3\times\mathbf{S}^2} b_2(\cos\theta)\,|v-v_*|^2 f(v'_*)d\sigma dv_* \\
&= 2\pi\left(\int_{\pi/2}^{\pi}\frac{b(\cos\theta)\sin^3\theta}{\sin^5(\theta/2)}d\theta + \int_0^{\pi/2}\frac{b(\cos\theta)\sin^3\theta}{\cos^5(\theta/2)}d\theta\right)\int_{\mathbf{R}^3}|v-v_*|^2 f(v_*)dv_* \\
&\le 2^{5/2}A_0(1+|v|^2)\int_{\mathbf{R}^3}(1+|v_*|^2)f(v_*)dv_*. \qquad \Box
\end{aligned}$$

Lemma 5. *For all* $0<R,\lambda<\infty$ *and all* $(s,x)\in\mathbf{R}^1\times\mathbf{R}^3$,

$$\int_{|v|\le R}\frac{\lambda\,dv}{\lambda+(s+x\cdot v)^2} \le \frac{4\pi^2R^3\lambda^{1/2}}{(\lambda+s^2+R^2|x|^2)^{1/2}}. \tag{31}$$

Proof. By scaling changes $\tilde{s}=s/\sqrt{\lambda}$, $\tilde{x}=Rx/\sqrt{\lambda}$ and $u=v/R$, the inequality (31) is equivalent to

$$\int_{|u|\le 1}\frac{du}{1+(\tilde{s}+\tilde{x}\cdot u)^2} \le \frac{4\pi^2}{(1+\tilde{s}^2+|\tilde{x}|^2)^{1/2}}, \quad (\tilde{s},\tilde{x})\in\mathbf{R}^1\times\mathbf{R}^3. \tag{32}$$

To prove (32) we can assume that $\tilde{s}^2 + |\tilde{x}|^2 > 1$. If $|\tilde{s}| \geq 2|\tilde{x}|$, then for all $|u| \leq 1$, $|\tilde{s} + \tilde{x} \cdot u| \geq |\tilde{s}| - |\tilde{x}| \geq \frac{1}{3}(|\tilde{s}| + |\tilde{x}|)$ which implies

$$\int_{|u|\leq 1} \frac{du}{1 + (\tilde{s} + \tilde{x} \cdot u)^2} \leq \frac{12\pi}{9 + \tilde{s}^2 + |\tilde{x}|^2} < \frac{4\pi^2}{(1 + \tilde{s}^2 + |\tilde{x}|^2)^{1/2}}.$$

If $|\tilde{s}| < 2|\tilde{x}|$, then $(1 + \tilde{s}^2 + |\tilde{x}|^2)^{1/2} \leq 4|\tilde{x}|$ and by changing variable we compute

$$\int_{|u|\leq 1} \frac{du}{1 + (\tilde{s} + \tilde{x} \cdot u)^2} \leq \pi \int_{-1}^{1} \frac{1}{1 + (\tilde{s} + |\tilde{x}|\, t)^2} dt \leq \frac{\pi^2}{|\tilde{x}|} \leq \frac{4\pi^2}{(1 + \tilde{s}^2 + |\tilde{x}|^2)^{1/2}}.$$

This proves the lemma. □

Let

$$\mathbf{K}^3 = \left\{ \left(\frac{2\pi}{T_1} l_1, \frac{2\pi}{T_2} l_2, \frac{2\pi}{T_3} l_3 \right) \middle| (l_1, l_2, l_2) \in \mathbf{Z}^3 \right\}, \quad |\mathbf{T}^3| = T_1 T_2 T_3.$$

Lemma 6. *Let $\phi(s, \mathrm{k})$ be a real or complex valued function on $\mathbf{R}^1 \times \mathbf{K}^3$ satisfying that $s \mapsto \phi(s, \mathrm{k})$ is measurable on $\mathbf{R}^1$ and*

$$\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} |\phi(s, \mathrm{k})|^p ds < \infty \quad \forall\, 1 \leq p \leq 2.$$

Let

$$\mathcal{F}[\phi](t, x) = \sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} \phi(s, \mathrm{k}) e^{-\mathrm{i}(st + \mathrm{k}\cdot x)} ds, \quad (t, x) \in \mathbf{R}^1 \times \mathbf{T}^3.$$

Then for any $1 < p < 2$ and $q = p/(p-1)$

$$\|\mathcal{F}[\phi]\|_{L^q(\mathbf{R}^1\times\mathbf{T}^3)} \leq \left(2\pi|\mathbf{T}^3|\right)^{1/q} \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} |\phi(s, \mathrm{k})|^p ds \right)^{1/p}. \tag{33}$$

Proof. (33) is in fact the Hausdorff–Young inequality or a special version of M.Riesz–Thorin convexity theorem: see Ref. 26, Chap. V, and consider measure spaces $(\mathbf{R}^1 \times \mathbf{K}^3, \mathcal{M}, \mu)$ and $(\mathbf{R}^1 \times \mathbf{T}^3, \mathcal{N}, \nu)$ where the σ-algebra $\mathcal{M}$ is defined by $E \in \mathcal{M} \Longleftrightarrow E^{\mathrm{k}} := \{s \in \mathbf{R}^1 \,|\, (s, \mathrm{k}) \in E\}$ is Lebesgue measurable for every $\mathrm{k} \in \mathbf{K}^3$, and the measure μ is then given by

$$\mu(E) = \sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} 1_E(s, \mathrm{k}) ds, \quad E \in \mathcal{M};$$

while in the second measure space, ν is the usual Lebesgue measure. We thus need only to check that inequality (33) holds for $p = 1$ and $p = 2$. For $p = 1$

(i.e. $q = \infty$), the inequality (33) is obvious, while for $p = 2 = q$, the equality sign in (33) holds due to the Parseval identity and Plancherel theorem. □

We next consider the L^1-compactness. Recall a criterion of the relative compactness in $L^1(\mathbf{R}^N)$(see e.g. Ref. 1): Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $L^1(\mathbf{R}^N)$. Then $\{f_n\}_{n=1}^{\infty}$ is relatively compact in $L^1(\mathbf{R}^N)$ if and only if $\{f_n\}_{n=1}^{\infty}$ satisfies

$$\sup_{n\geq 1} \|f_n\|_{L^1(\mathbf{R}^N)} < \infty, \quad \sup_{n\geq 1} \int_{|z|>R} |f_n(z)|dz \to 0 \quad (R \to \infty),$$

and

$$\sup_{n\geq 1} \|f_n(\cdot + h) - f_n\|_{L^1(\mathbf{R}^N)} \to 0 \quad (|h| \to 0).$$

Lemma 7. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $L^1(\mathbf{R}^1 \times \mathbf{R}^3)$ satisfying that the functions $x \mapsto F_n(t,x)$ are periodic with the period $\mathbf{T} = (T_1, T_2, T_3)$ and for some $0 < T < \infty$,* $\operatorname{supp} F_n \subset [0, T] \times \mathbf{R}^3$ $(\forall\, n \geq 1)$. *Suppose further that*

$$\sup_{n\geq 1} \|F_n\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)} < \infty,$$

$$\sup_{n1} \|F_n(\cdot + \tau, \cdot + h) - F_n\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)} \to 0 \quad (|\tau| + |h| \to 0).$$

Then $\{F_n\}_{n=1}^{\infty}$ is relatively compact in $L^1(\mathbf{R}^1 \times \mathbf{T}^3)$.

This lemma can be reduced to a special case of the above criterion of the compactness in $L^1(\mathbf{R}^4)$ by using the following property: Let $F(x)$ be a measurable and periodic function on $\mathbf{R}^3$ with the period $\mathbf{T}$, and let (for instance) $\rho(x) = e^{-\alpha|x|^2}$, $\alpha > 0$. Then

$$\int_{\mathbf{R}^3} |F(x)|\rho(x)dx = \int_{\mathbf{T}^3} |F(x)| \left(\sum_{\mathrm{m}\in\mathbf{Z}^3} \rho(x + \mathrm{m}\mathbf{T}) \right) dx \leq C \int_{\mathbf{T}^3} |F(x)|dx$$

where $\mathrm{m}\mathbf{T} := (m_1T_1, m_2T_2, m_3T_3)$ and $C < \infty$ depends only on $\mathbf{T}$ and α. Then using this property to $f_n(t,x) = F_n(t,x)e^{-|x|^2}$ it is easily checked that $\{f_n\}_{n=1}^{\infty}$ is relatively compact in $L^1(\mathbf{R}^4)$ and therefore $\{F_n\}_{n=1}^{\infty}$ is relatively compact in $L^1(\mathbf{R}^1 \times \mathbf{T}^3)$.

3. AVERAGING COMPACTNESS OF WEAK SOLUTIONS

In our proof of the velocity-averaging compactness of weak solutions we will use the following functions:

$$\beta_{s,\mathrm{k}}(t) = \frac{t}{(1 + s^2 + |\mathrm{k}|^2)^{\delta} + t^2}, \quad \alpha_{s,\mathrm{k}}(t) = 1 - t\beta_{s,\mathrm{k}}(t)$$

where $\delta > 0$ is a constant. In our estimates below for averaging compactness we shall choose $1/2 << \delta < 1$. Let us list some properties of $\alpha_{s,\mathrm{k}}$, $\beta_{s,\mathrm{k}}$:

$$\alpha_{s,\mathrm{k}}(s+\mathrm{k}\cdot v) + (s+\mathrm{k}\cdot v)\beta_{s,\mathrm{k}}(s+\mathrm{k}\cdot v) \equiv 1, \tag{34}$$

$$0 \le \alpha_{s,\mathrm{k}}(t) = \frac{(1+s^2+\mathrm{k}|^2)^\delta}{(1+s^2+|\mathrm{k}|^2)^\delta + t^2} \le 1, \tag{35}$$

$$|\beta_{s,\mathrm{k}}(t)| = \frac{|t|}{(1+s^2+|\mathrm{k}|^2)^\delta + t^2} \le \frac{1}{2}(1+s^2+|\mathrm{k}|^2)^{-\delta/2}, \tag{36}$$

$$\int_{|v|\le R} \alpha_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)dv \le 4\pi^2 R^3\,(1+s^2+|\mathrm{k}|^2)^{-(1-\delta)/2}. \tag{37}$$

Here the last inequality (37) is due to Lemma 5 with $\lambda = (1+s^2+|\mathrm{k}|^2)^\delta$ and $R1$. Let

$$\beta^{(1)}_{s,\mathrm{k}}(t) = \frac{d}{dt}\beta_{s,\mathrm{k}}(t).$$

Then

$$\left|\beta^{(1)}_{s,\mathrm{k}}(t)\right| = \frac{|(1+s^2+|\mathrm{k}|^2)^\delta - t^2|}{((1+s^2+|\mathrm{k}|^2)^\delta + t^2)^2} \le \frac{1}{(1+s^2+|\mathrm{k}|^2)^\delta}$$

which gives

$$|\nabla_v(\beta_{s,\mathrm{k}}(s+\mathrm{k}\cdot v))| = |\beta^{(1)}_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)|\,|\mathrm{k}| \le (1+s^2+|\mathrm{k}|^2)^{-(\delta-1/2)}.$$

This together with (36) implies that

$$\sup_{v\in\mathbf{R}^3}\{|\beta_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)| + |\nabla_v(\beta_{s,\mathrm{k}}(s+\mathrm{k}\cdot v))|\} \le \frac{3}{2}(1+s^2+|\mathrm{k}|^2)^{-(\delta-1/2)}. \tag{38}$$

For any $F \in L^2(\mathbf{R}^1\times\mathbf{T}^3)\cap L^1(\mathbf{R}^1\times\mathbf{T}^3)$, the Fourier transform of F is defined by

$$\widehat{F}(s,\mathrm{k}) = \int_{\mathbf{R}^1\times\mathbf{T}^3} F(t,x)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dxdt.$$

By Parseval identity and Plancherel theorem we have

$$\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|\widehat{F}(s,\mathrm{k})|^2 ds = 2\pi|\mathbf{T}^3|\int_{\mathbf{R}^1\times\mathbf{T}^3}|F(t,x)|^2 dxdt. \tag{39}$$

Let $\langle g\rangle$ be the velocity average function as given in (22), i.e.

$$\langle g\rangle(t,x) = \int_{\mathbf{R}^3} g(t,x,v)dv, \quad (t,x)\in\mathbf{R}^1\times\mathbf{T}^3.$$

The Fourier transform of $\langle g\rangle$ is then given by

$$\langle g\rangle\widehat{\ }(s,\mathrm{k})=\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} g(t,x,v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt. \tag{40}$$

Introduce

$$\|F\|_{H^\eta(\mathbf{R}^1\times\mathbf{T}^3)}=\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(1+s^2+|\mathrm{k}|^2)^\eta|\widehat{F}(s,\mathrm{k})|^2ds\right)^{1/2},\quad \eta>0,$$

$$\|\psi\|_{1,\infty}=\sup_{v\in\mathbf{R}^3}\{|\psi(v)|+|\nabla\psi(v)|\},\quad \|\zeta\|_{1,\infty}=\sup_{t\in\mathbf{R}^1}\left\{|\zeta(t)|+|\frac{d}{dt}\zeta(t)|\right\}.$$

Our first main result of this paper can be stated as follows:

Theorem 1. *Let B be a collision kernel satisfying* (6) – (7) *and let $f(t,x,v)$ be a weak solution to Eqs. (1)–(3) with initial datum $f_0\in L^1_2(\mathbf{T}^3\times\mathbf{R}^3)$ satisfying* $\sup_{t\ge0}\|f(t)\|_{L^1_2(\mathbf{T}^3\times\mathbf{R}^3)}<\infty$. *Assume that f satisfy the entropy inequality (15). Then there exists a constant $0<\eta<1$ depending only on γ such that for any $0<T<\infty$, $1<R<\infty$ and any $\psi\in C^1(\mathbf{R}^3)$, $\zeta\in C^1(\mathbf{R}^1)$ with* $\operatorname{supp}\psi\subset\{v\in\mathbf{R}^3\mid |v|\le R\}$, $\operatorname{supp}\zeta\subset(0,T)$*, we have*

$$\|\langle f\zeta\psi\rangle\|_{H^\eta(\mathbf{R}^1\times\mathbf{T}^3)}\le C\|\zeta\|_{1,\infty}\|\psi\|_{1,\infty} \tag{41}$$

and consequently for all $\tau\in\mathbf{R}^1$, $h\in\mathbf{R}^3$

$$\|\langle f\zeta\psi\rangle(\cdot+\tau,\ \cdot+h)-\langle f\zeta\psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}\le C\ \|\zeta\|_{1,\infty}\|\psi\|_{1,\infty}(\tau^2+|h|^2)^{\eta/2} \tag{42}$$

where $C=(1+A_0)^{1/2}C_{\mathcal{K}_0,R,T}<\infty$ and $C_{\mathcal{K}_0,R,T}$ depends only on $\mathcal{K}_0$, R, T and on γ, $|\mathbf{T}^3|$. Here $\mathcal{K}_0$ is any given constant satisfying $\sup_{t\ge0}\|f(t)\|_{L^1_2(\mathbf{T}^3\times\mathbf{R}^3)}\le\mathcal{K}_0<\infty$.

Proof. We can assume that $\zeta(t)\not\equiv0$, $\psi(v)\not\equiv0$. Also by replacing $\zeta(t)$ and $\psi(v)$ with $\zeta(t)/\|\zeta\|_{1,\infty}$ and $\psi(v)/\|\psi\|_{1,\infty}$ we can assume that $\|\zeta\|_{1,\infty}=\|\psi\|_{1,\infty}=1$. In the following we denote by $C_{\{*,*,*,\dots\}}$ the positive and finite constants that depend only on its arguments $*$, $*$, ... and may depend on γ and $|\mathbf{T}^3|$. For convenience of our derivation we extend $f(t)$ on $t\in\mathbf{R}^1$ as follows:

$$f(t,x,v)=f_0(x,v)\quad\text{for}\quad t<0.$$

Proof of (41). We will use the functions $\alpha_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)$, $\beta_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)$ introduced above. In view of identity (34) we consider the following decomposition:

$$\psi(v)=\Phi_{s,\mathrm{k}}(v)+\Psi_{s,\mathrm{k}}(v)(s+\mathrm{k}\cdot v)$$

where

$$\Phi_{s,\mathrm{k}}(v) = \psi(v)\alpha_{s,\mathrm{k}}(s + \mathrm{k}\cdot v), \quad \Psi_{s,\mathrm{k}}(v) = \psi(v)\beta_{s,\mathrm{k}}(s + \mathrm{k}\cdot v).$$

According to (40) this gives

$$\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k}) = \int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta(t)\Phi_{s,\mathrm{k}}(v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$+ \int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta(t)\Psi_{s,\mathrm{k}}(v)(s+\mathrm{k}\cdot v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt. \tag{43}$$

Applying the weak form (10) with the test function

$$\varphi(t,x,v) = \zeta(t)\Psi_{s,\mathrm{k}}(v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}$$

and noting that $\zeta(T) = \zeta(0) = 0$ we compute with $\zeta_1(t) = \frac{d}{dt}\zeta(t)$

$$0 = \int_0^T \int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta_1(t)\Psi_{s,\mathrm{k}}(v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$-\mathrm{i}\int_0^T \int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta(t)\Psi_{s,\mathrm{k}}(v)(s+\mathrm{k}\cdot v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$+\frac{1}{4}\int_0^T \int_{\mathbf{T}^3\times\mathbf{R}^3} \zeta(t)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}Q_B(f\mid\Delta\Psi_{s,\mathrm{k}})dvdxdt$$

and so (using $\operatorname{supp}\zeta \subset (0,T)$ again)

$$\mathrm{i}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta(t)\Psi_{s,\mathrm{k}}(v)(s+\mathrm{k}\cdot v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$= \int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta_1(t)\Psi_{s,\mathrm{k}}(v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$+\frac{1}{4}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} \zeta(t)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}Q_B(f\mid\Delta\Psi_{s,\mathrm{k}})dvdxdt.$$

From this we see that the Eq. (43) becomes

$$\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k}) = \int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta(t)\Phi_{s,\mathrm{k}}(v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$-\mathrm{i}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)\zeta_1(t)\Psi_{s,\mathrm{k}}(v)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dvdxdt$$
$$-\frac{\mathrm{i}}{4}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} \zeta(t)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}Q_B(f\mid\Delta\Psi_{s,\mathrm{k}})dvdxdt. \tag{44}$$

Let us denote

$$\varrho(s,\mathrm{k}) = (1+s^2+|\mathrm{k}|^2)^{1/2}. \tag{45}$$

For the sake of integrability we consider, for every $0<N<\infty$,

$$F_N(s,\mathrm{k}) = (\varrho(s,\mathrm{k}))^{\eta}\, \overline{\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k})}\, 1_{\{|s|\le N\}}\, 1_{\{|\mathrm{k}|\le N\}} \tag{46}$$

where the value of $\eta>0$ (depending only on γ) will be given later. It is easily checked that

$$\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds = \sum_{|\mathrm{k}|\le N}\int_{|s|\le N}(\varrho(s,\mathrm{k}))^{2\eta}|\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k})|^2 ds < \infty,$$

$$\lim_{N\to\infty}\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds = \sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{2\eta}|\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k})|^2 ds.$$

Therefore to prove (41) it suffices to prove that

$$\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds \le (1+A_0)C_{\mathcal{K}_0,R,T} \qquad \forall\, 0<N<\infty. \tag{47}$$

Let

$$\widehat{f\zeta}(s,\mathrm{k})_v = \int_{\mathbf{R}^1\times\mathbf{T}^3} f(t,x,v)\zeta(t)e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}dtdx,$$

$$\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k}) = \Delta\Psi_{s,\mathrm{k}}(v,v_*,v',v'_*)\,(\varrho(s,\mathrm{k}))^{\eta}F_N(s,\mathrm{k}).$$

We compute using (46) and (44)

$$\begin{aligned}
\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds &= \sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{\eta}F_N(s,\mathrm{k})\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k})ds \\
&= \sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{\eta}F_N(s,\mathrm{k})\left[\int_{\mathbf{R}^3}\Phi_{s,\mathrm{k}}(v)\widehat{f\zeta}(s,\mathrm{k})_v dv\right]ds \\
&\quad -\mathrm{i}\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{\eta}F_N(s,\mathrm{k})\left[\int_{\mathbf{R}^3}\Psi_{s,\mathrm{k}}(v)\widehat{f\zeta_1}(s,\mathrm{k})_v dv\right]ds \\
&\quad -\frac{\mathrm{i}}{4}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B(v-v_*,\sigma)\zeta(t)\Pi_{\mathrm{F}}(f)\mathcal{F}[\phi_{(v,v_*,v',v'_*)}](t,x)d\mu \\
&:= I_1+I_2+I_3.
\end{aligned}$$

Here and below, $d\mu = d\sigma dv_* dv dx dt$,

$$\mathcal{F}\left[\phi_{(v,v_*,v',v'_*)}\right](t,x) = \sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})e^{-\mathrm{i}(st+\mathrm{k}\cdot x)}ds.$$

□

Estimates of I_1 and I_2

By Cauchy–Schwarz inequality and recalling that $|\psi(v)| \leq 1_{\{|v|\leq R\}}$, $|\alpha_{s,\mathrm{k}}(s + \mathrm{k}\cdot v)| \leq 1$ (see (35)) and (37) we have

$$\int_{\mathbf{R}^3} |\Phi_{s,\mathrm{k}}(v)| \left|\widehat{f\zeta}(s,\mathrm{k})_v\right| dv$$
$$\leq \left(\int_{\mathbf{R}^3} |\psi(v)|^2 |\alpha_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)|^2 dv\right)^{1/2} \left(\int_{\mathbf{R}^3} |\widehat{f\zeta}(s,\mathrm{k})_v|^2 dv\right)^{1/2}$$
$$\leq C_R(\varrho(s,\mathrm{k}))^{-(1-\delta)/2} \left(\int_{\mathbf{R}^3} |\widehat{f\zeta}(s,\mathrm{k})_v|^2 dv\right)^{1/2}.$$

If we choose $0 < \eta \leq (1-\delta)/2$, then

$$|I_1| \leq C_R \sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} (\varrho(s,\mathrm{k}))^{\eta} |F_N(s,\mathrm{k})| (\varrho(s,\mathrm{k}))^{-(1-\delta)/2} \left(\int_{\mathbf{R}^3} |\widehat{f\zeta}(s,\mathrm{k})_v|^2 dv\right)^{1/2} ds$$
$$\leq C_R \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} |F_N(s,\mathrm{k})|^2 ds\right)^{1/2} \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1}\int_{\mathbf{R}^3} |\widehat{f\zeta}(s,\mathrm{k})_v|^2 dv ds\right)^{1/2}.$$

Since, by identity (39),

$$\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1}\int_{\mathbf{R}^3} |\widehat{f\zeta}(s,\mathrm{k})_v|^2 dv ds = 2\pi|\mathbf{T}^3| \int_{\mathbf{R}^3}\int_{\mathbf{R}^1\times\mathbf{T}^3} |\zeta(t) f(t,x,v)|^2 dx dt dv$$
$$\leq 2\pi|\mathbf{T}^3| \int_0^T dt \int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v) dv dx \leq 2\pi|\mathbf{T}^3|\mathcal{K}_0 T$$

it follows that

$$|I_1| \leq C_{\mathcal{K}_0,R,T} \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} |F_N(s,\mathrm{k})|^2 ds\right)^{1/2}.$$

Similarly we use $|\Psi_{s,\mathrm{k}}(v)| = |\psi(v)||\beta_{s,\mathrm{k}}(s+\mathrm{k}\cdot v)| \leq \frac{1}{2}|\psi(v)|(\varrho(s,\mathrm{k}))^{-\delta}$ (see (36)) and choose η satisfying also $0 < \eta \leq \delta$. Then

$$|I_2| \leq C_R \sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} (\varrho(s,\mathrm{k}))^{\eta} |F_N(s,\mathrm{k})| (\varrho(s,\mathrm{k}))^{-\delta} \left(\int_{\mathbf{R}^3} |\widehat{f\zeta_1}(s,\mathrm{k})_v|^2 dv\right)^{1/2} ds$$
$$\leq \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} |F_N(s,\mathrm{k})|^2 ds\right)^{1/2} \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1}\int_{\mathbf{R}^3} |\widehat{f\zeta_1}(s,\mathrm{k})_v|^2 dv ds\right)^{1/2}$$
$$\leq C_{\mathcal{K}_0,R,T} \left(\sum_{\mathrm{k}\in\mathbf{K}^3} \int_{\mathbf{R}^1} |F_N(s,\mathrm{k})|^2 ds\right)^{1/2}.$$

Estimate of I_3

As shown in the Introduction we have (write $\Pi_{\mathrm{F}}(f) = \Pi_2(f)\Pi_1(f)$)

$$
\begin{aligned}
|I_3| &\le \frac{1}{4}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\zeta(t)||\Pi_{\mathrm{F}}(f)|\left|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}](t,x)\right| d\mu \\
&\le \left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\zeta(t)||\Pi_2(f)|^2 d\mu\right)^{1/2} \\
&\quad\times\left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\zeta(t)||\Pi_1(f)|^2|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^2 d\mu\right)^{1/2}
\end{aligned}
$$

and, recalling that $\operatorname{supp}\zeta \subset (0, T)$, $|\zeta(t)| \le 1$ and using (20), (15) and (13),

$$
\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\zeta(t)||\Pi_2(f)|^2 d\mu \le \int_0^T dt\int_{\mathbf{T}^3} D(f)(t,x)dx \le C_{\mathcal{K}_0}.
$$

Therefore we obtain

$$
|I_3| \le C_{\mathcal{K}_0}\left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\zeta(t)||\Pi_1(f)|^2|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^2 d\mu\right)^{1/2}. \tag{48}
$$

Since $\operatorname{supp}\psi \subset \{v \in \mathbf{R}^3 \mid |v| \le R\}$, this gives (by definition of $\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})$)

$$
\begin{aligned}
&|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}](t,x)|^2 \\
&\quad\le |\mathcal{F}\left[\phi_{(v,v_*,v',v'_*)}\right](t,x)|^2\left(1_{\{|v|\le R\}} + 1_{\{|v_*|\le R\}} + 1_{\{|v'|\le R\}} + 1_{\{|v'_*|\le R\}}\right).
\end{aligned}
$$

Again by definition of $\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})$, the function $|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}](t,x)|$ is invariant under the changes $v \leftrightarrow v_*$ and $(v, v_*) \leftrightarrow (v', v'_*)$. Thus

$$
\begin{aligned}
&\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,|\zeta(t)||\Pi_1(f)|^2|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^2 d\mu \\
&\quad\le 4\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\,1_{\{|v|\le R\}}|\zeta(t)||\Pi_1(f)|^2|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^2 d\mu.
\end{aligned}
$$

Next let

$$
\lambda = \lambda(|v-v_*|) = \begin{cases} |\gamma| - 2, & |v-v_*| \le 1; \\ |\gamma| + 2, & |v-v_*| > 1. \end{cases} \tag{49}
$$

and let $2 < p < \infty$, $q = p/(p-1)$. Here and below p dependsonly on $|\gamma|$. We then compute using Hölder inequality

$$\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|\zeta(t)||\Pi_1(f)|^2|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^2 d\mu$$

$$\le \left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{\lambda}(\sin\theta)^2|\zeta(t)||\Pi_1(f)|^{2p} d\mu\right)^{1/p}$$

$$\times\left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{-\lambda q/p}(\sin\theta)^{-2q/p}|\zeta(t)||\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^{2q} d\mu\right)^{1/q}. \quad (50)$$

For the first factor we use inequality $|\Pi_1(f)|^{2p} \le 2^{2p-1}(ff_* + f'f'_*)$ (because $0 \le f \le 1$) to get

$$\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{\lambda}\sin^2\theta|\zeta(t)||\Pi_1(f)|^{2p} d\mu$$

$$\le 2^{2p-1}\int_0^T\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}1_{\{|v-v_*|\le 1\}}|v-v_*|^{|\gamma|-2}\sin^2\theta(ff_*+f'f'_*)d\mu$$

$$+2^{2p-1}\int_0^T\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}1_{\{|v-v_*|>1\}}|v-v_*|^{|\gamma|+2}\sin^2\theta(ff_*+f'f'_*)d\mu$$

$$:= 2^{2p-1}J_{|\gamma|-2} + 2^{2p-1}J_{|\gamma|+2}.$$

For the $J_{|\gamma|-2}$ term we use Lemma 2 to obtain

$$J_{|\gamma|-2} \le 2A_0\int_0^T dt\int_{\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{R}^3}\frac{f(t,x,v)f(t,x,v_*)}{|v-v_*|^2}dv dv_* dx$$

$$\le C_0A_0\int_0^T dt\int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v)|v|dvdx \le C_0A_0\mathcal{K}_0T.$$

For the $J_{|\gamma|+2}$ term we compute

$$J_{|\gamma|+2} \le J^{(1)}_{|\gamma|+2} + J^{(2)}_{|\gamma|+2}$$

with

$$J^{(1)}_{|\gamma|+2} := \int_0^T\int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{|v|\le R\}}|v-v_*|^{|\gamma|+2}\sin^2\theta\, ff_* d\mu$$

$$\le A_0C_0R^5\int_0^T dt\int_{\mathbf{T}^3\times\mathbf{R}^3} f_*(1+|v_*|^2)dv_*dx \le C_0A_0R^5\mathcal{K}_0T$$

and, using Lemma 4,

$$\begin{aligned}
J^{(2)}_{|\gamma|+2} &:= \int_0^T \int_{\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{|v|\le R\}}|v-v_*|^{|\gamma|+2}\sin^2\theta\, f' f'_* d\mu \\
&= \int_0^T \int_{\mathbf{T}^3}\int_{|v|\le R}\left(\int_{\mathbf{R}^3\times\mathbf{S}^2} B\,\sin^2\theta\,|v-v_*|^{|\gamma|+2} f' f'_* d\sigma dv_*\right) dv dx dt \\
&\le C_0 A_0 \int_0^T \int_{\mathbf{T}^3}\int_{|v|\le R}(1+|v|^2)\left(\int_{\mathbf{R}^3} f_*(1+|v_*|^2)dv_*\right) dv dx dt \\
&\le A_0 C_0 R^5 \int_0^T dt \int_{\mathbf{T}^3\times\mathbf{R}^3} f(t,x,v_*)(1+|v_*|^2)dv_* dx \le C_0 A_0 R^5 \mathcal{K}_0 T.
\end{aligned}$$

Summarizing we obtain that the first factor in (50) is bounded:

$$\left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{\lambda}\sin^2\theta|\zeta(t)||\Pi_1(f)|^{2p}d\mu\right)^{1/p} \le A_0^{1/p} C_{\mathcal{K}_0,R,T}. \tag{51}$$

To estimate the second factor in (50), let

$$p_1 = \frac{2q}{2q-1}\quad\left(=\frac{2p}{p+1}<2\right).$$

Then by the Hausdorff–Young inequality (Lemma 6) we have

$$\int_{\mathbf{R}^1\times\mathbf{T}^3}|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}](t,x)|^{2q}dt dx \le 2\pi|\mathbf{T}^3|\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})|^{p_1}ds\right)^{2q/p_1}.$$

Hence by $|\zeta(t)|\le 1$ we get

$$\begin{aligned}
&\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{-\lambda q/p}(\sin\theta)^{-2q/p}|\zeta(t)||\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^{2q}d\mu \\
&\le \int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{-\lambda q/p}(\sin\theta)^{-2q/p} \\
&\quad\times\left(\int_{\mathbf{R}^1\times\mathbf{T}^3}|\mathcal{F}[\phi_{(v,v_*,v',v'_*)}](t,x)|^{2q}dt dx\right) d\sigma dv_* dv \\
&\le 2\pi|\mathbf{T}^3|\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{-\lambda q/p}(\sin\theta)^{-2q/p} \\
&\quad\times\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})|^{p_1}ds\right)^{2q/p_1} d\sigma dv_* dv.
\end{aligned} \tag{52}$$

Further estimate: By definition of $\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})$ and $\Psi_{s,\mathrm{k}}(v)$, and applying (38) and Lemma 1 we compute

$$|\phi_{(v,v_*,v',v'_*)}(s,\mathrm{k})| \le C_0|v-v_*|\sin\theta\,(\varrho(s,\mathrm{k}))^{-(2\delta-1-\eta)}|F_N(s,\mathrm{k})|.$$

This gives (because$(\sin\theta)^{-2q/p}(\sin\theta)^{2q} = \sin^2\theta$)

$$(52) \le 2\pi|\mathbf{T}^3|C_0^{2q}\left(\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\,1_{\{|v|\le R\}}|v-v_*|^{-q(\lambda/p-2)}\sin^2\theta d\sigma dv_* dv\right)$$
$$\times\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{-p_1(2\delta-1-\eta)}|F_N(s,\mathrm{k})|^{p_1}ds\right)^{2q/p_1}. \tag{53}$$

We first estimate the second integral in the right hand side of (53). Let

$$p_2 = \frac{2}{2-p_1}\,(=p+1),\quad q_2 = \frac{p_2}{p_2-1}\left(=\frac{2}{p_1}\right).$$

Then $p_1p_2 = 2p,\ p_1q_2 = 2$. So by Hölder inequality with index (p_2, q_2) we get

$$\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{-p_1(2\delta-1-\eta)}|F_N(s,\mathrm{k})|^{p_1}ds\right)^{2q/p_1}$$
$$\le\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{-2p(2\delta-1-\eta)}ds\right)^{q/p}\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2ds\right)^{q}.$$

Now we choose $2 < p < \infty$ that depends only γ, satisfying

$$\eta_1 := q\left(\frac{|\gamma|-2}{p}-2\right)+|\gamma| < 3,\quad \eta_2 := q\left(\frac{|\gamma|+2}{p}-2\right)+|\gamma| > 3$$

which is equivalent to

$$\max\left\{2,\ \frac{1}{5-|\gamma|}\right\} < p < \frac{5}{5-|\gamma|}.$$

We then choose $\delta > 0$, $\eta > 0$ such that $1/2 + 1/p < \delta < 1$ and

$$0 < \eta < \min\left\{2\delta - 1 - \frac{2}{p},\ \frac{1-\delta}{2},\ \delta\right\}.$$

Then the above estimates hold for this number η. And we have

$$\beta := 2p\,(2\delta - 1 - \eta) > 4$$

which implies (by definition of ϱ in (45)) that

$$\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}(\varrho(s,\mathrm{k}))^{-\beta}ds \le C_\beta\int_{\mathbf{R}^4}(1+|y|^2)^{-\beta/2}dy < \infty.$$

Next we compute the first integral in the right hand side of (53): By definition of λ (see (49)) and η_1, η_2 we have

$$\begin{aligned}
&\int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{-q(\lambda/p-2)}\sin^2\theta d\sigma dv_* dv \\
&= \int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}1_{\{|v-v_*|\le 1\}}|v-v_*|^{-\eta_1+|\gamma|}\sin^2\theta d\sigma dv_* dv \\
&\quad + \int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}1_{\{|v-v_*|> 1\}}|v-v_*|^{-\eta_2+|\gamma|}\sin^2\theta d\sigma dv_* dv \\
&\le A_0\int_{|v|\le R}\int_{\mathbf{R}^3}\left(1_{\{|v-v_*|\le 1\}}|v-v_*|^{-\eta_1} + 1_{\{|v-v_*|>1\}}|v-v_*|^{-\eta_2}\right)dv_* dv \\
&\le A_0 C_0 R^3\left(\int_{|z|\le 1}|z|^{-\eta_1}dz + \int_{|z|>1}|z|^{-\eta_2}dz\right) = A_0 C_R < \infty.
\end{aligned}$$

Summarizing above we get

$$(53) \le A_0 C_R\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds\right)^q. \tag{54}$$

Combining (52)–(54) we obtain the following estimate for the second factor in (50):

$$\begin{aligned}
&\left(\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^6\times\mathbf{S}^2} B\, 1_{\{|v|\le R\}}|v-v_*|^{-\lambda q/p}(\sin\theta)^{-2q/p}|\zeta(t)||\mathcal{F}[\phi_{(v,v_*,v',v'_*)}]|^{2q}d\mu\right)^{1/q} \\
&\le A_0^{1/q}C_{\mathcal{K}_0,R,T}\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds.
\end{aligned}$$

This together with (48), (50), and (51) gives

$$|I_3| \le A_0^{1/2}\, C_{\mathcal{K}_0,R,T}\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds\right)^{1/2}.$$

Summarizing the above estimates for I_i ($i = 1, 2, 3$) we obtain for all $0 < N < \infty$

$$\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds \le (1 + A_0^{1/2})C_{\mathcal{K}_0,R,T}\left(\sum_{\mathrm{k}\in\mathbf{K}^3}\int_{\mathbf{R}^1}|F_N(s,\mathrm{k})|^2 ds\right)^{1/2}$$

which implies (47) and (41) is proven.

Proof of (42). Recalling our assumption on ζ, ψ and supp $\zeta \subset (0, T)$ we have

$$\|\langle f\zeta\psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)} \le T \sup_{t0} \|f(t)\|_{L^1(\mathbf{T}^3\times\mathbf{R}^3)} \le T\mathcal{K}_0.$$

Therefore to prove (42) we can assume that $|\tau| \le 1$. This implies that the functions $t \mapsto \langle f\zeta\psi\rangle(t+\tau,\ x+h)$ are supported on $[-1, T+1]$. By Cauchy-Schwarz inequality we then obtain

$$\begin{aligned}&\|\langle f\zeta\psi\rangle(\cdot+\tau,\ \cdot+h) - \langle f\zeta\psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}\\ &\le \sqrt{(T+2)|\mathbf{T}^3|}\ \|\langle f\zeta\psi\rangle(\cdot+\tau,\ \cdot+h) - \langle f\zeta\psi\rangle\|_{L^2(\mathbf{R}^1\times\mathbf{T}^3)}.\end{aligned}$$

By $0 < \eta < 1$ it is easy to show that

$$|e^{\mathrm{i}(s\tau+\mathrm{k}\cdot h)} - 1|^2 \le 4(s^2+|\mathrm{k}|^2)^\eta(\tau^2+|h|^2)^\eta$$

from which we deduce (using (39))

$$\begin{aligned}&\|\langle f\zeta\psi\rangle(\cdot+\tau,\ \cdot+h) - \langle f\zeta\psi\rangle\|^2_{L^2(\mathbf{R}^1\times\mathbf{R}^3)}\\ &= \frac{1}{2\pi|\mathbf{T}^3|}\sum_{\mathrm{k}}\int_{\mathbf{R}^1} |\langle f\zeta\psi\rangle\widehat{\ }(s,\mathrm{k})|^2|e^{\mathrm{i}(s\tau+\mathrm{k}\cdot h)}-1|^2 ds\\ &\le \frac{4}{2\pi|\mathbf{T}^3|}\|\langle f\zeta\psi\rangle\|^2_{H^\eta(\mathbf{T}^3\times\mathbf{R}^3)}(\tau^2+|h|^2)^\eta \le C^2(\tau^2+|h|^2)^\eta.\end{aligned}$$

This proves (42) and the proof of the theorem is complete. □

Now applying the above estimate (42) we can prove the following averaging compactness of weak (approximate) solutions of Eqs. (1)–(3).

Theorem 2. *Let B, B_n be collision kernels with B satisfying (6)–(7) and $0 \le B_n \le B$ $(n = 1, 2, ...)$. Let f^n be weak solutions of Eqs. (1)–(3) with the kernel B_n and the initial data $f^n|_{t=0} = f^n_0$ satisfying $\sup_{n\ge1}\sup_{t\ge0} \|f^n(t)\|_{L^1_2} < \infty$. Assume also that f^n satisfy the entropy inequality (15) (corresponding to B_n). Then for any $T \in (0,\infty)$ and any $\Psi \in L^\infty([0,T]\times\mathbf{T}^3\times\mathbf{R}^3)$, the set $\{\langle f^n\Psi\rangle \mid n = 1, 2, 3, ...\}$ is relatively compact in $L^1([0,T]\times\mathbf{T}^3)$.*

Proof. Let $\mathcal{K}_0 = \sup_{n\ge1}\sup_{t\ge0}\|f^n(t)\|_{L^1_2}$. As did before we define $f^n(t,x,v) = f^n_0(x,v)$ for $t < 0$. Let $0 < T < \infty$ and $\Psi \in L^\infty([0,T]\times\mathbf{T}^3\times\mathbf{R}^3)$ be given. We extend Ψ on $\mathbf{R}^1\times\mathbf{R}^3\times\mathbf{R}^3$ in the way that the extension Ψ is periodic in x with the period $\mathbf{T}$ and $\Psi(t,\cdot,\cdot) = 0$ for $t \in \mathbf{R}^1 \setminus [0,T]$. This implies that supp$\langle f^n\Psi\rangle \subset [0,T]\times\mathbf{R}^3$ and $\sup_{n\ge1}\|\langle f^n\Psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)} \le \|\Psi\|_\infty T\mathcal{K}_0$ where $\|\Psi\|_\infty = \|\Psi\|_{L^\infty(\mathbf{R}^1\times\mathbf{R}^3\times\mathbf{R}^3)}$. Therefore, by Lemma 7, to prove the relative compactness of $\{\langle f^n\Psi\rangle\}_{n=1}^{\infty}$ in $L^1(\mathbf{R}^1\times\mathbf{T}^3)$ we need only to prove that

$$\lim_{|\tau|+|h|\to 0}\sup_{n\ge 1}\|\langle f^n\Psi\rangle(\cdot+\tau,\ \cdot+h)-\langle f^n\Psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}=0. \tag{55}$$

For any $1<R<\infty$, the L^1_2-bounds of $\{f^n(t)\}_{n=1}^{\infty}$ implies

$$\sup_{n\ge 1}\int_{\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3} f^n(t,x,v)|\Psi(t,x,v)|1_{\{|v|>R\}}dvdxdt\le\|\Psi\|_{\infty}T\mathcal{K}_0\frac{1}{R^2}.$$

Thus for notation convenience we can assume further that Ψ has been truncated in the v-variable: $\Psi(\cdot,\cdot,v)=0$ for all $|v|>R$, so that $\Psi\in L^1(\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3)$.

Now given any $0<\varepsilon<\min\{1,T/4\}$. Choose a function $\zeta_\varepsilon\in C_c^{\infty}(\mathbf{R}^1)$ satisfying supp $\zeta_\varepsilon\subset(0,T)$, $0\le\zeta_\varepsilon(t)\le 1$ on $\mathbf{R}^1$, and $\zeta_\varepsilon(t)=1$ $\forall t\in[2\varepsilon,T-2\varepsilon]$. Also choose a function $\chi\in C_c^{\infty}(\mathbf{R}^3)$ satisfying supp$\chi\subset\{z\in\mathbf{R}^3\mid |z|\le 1\}$, $0\le\chi(z)\le 1$ on $z\in\mathbf{R}^3$, and $\int_{\mathbf{R}^3}\chi(z)dz=1$. Let $\chi_\varepsilon(z)=\varepsilon^{-3}\chi(z/\varepsilon)$,

$$\Psi_\varepsilon(t,x,v)=\zeta_\varepsilon(t)\int_{\mathbf{R}^3}\Psi(t,x,z)\chi_\varepsilon(v-z)dz=\zeta_\varepsilon(t)\int_{\mathbf{R}^3}\Psi(t,x,v-\varepsilon z)\chi(z)dz.$$

It is obvious that Ψ_ε is still periodic in x with the period $\mathbf{T}$. We compute

$$\begin{aligned}&\|\langle f^n\Psi\rangle(\cdot+\tau,\ \cdot+h)-\langle f^n\Psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}\\&\quad\le 2\|\langle f^n(\Psi-\Psi_\varepsilon)\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}+\|\langle f^n\Psi_\varepsilon\rangle(\cdot+\tau,\ \cdot+h)-\langle f^n\Psi_\varepsilon\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}.\end{aligned} \tag{56}$$

By $\Psi(t,\cdot,\cdot)=\Psi_\varepsilon(t,\cdot,\cdot)=0$ for all $t\in\mathbf{R}^1\setminus[0,T]$ we have

$$\|\langle f^n(\Psi-\Psi_\varepsilon)\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}\le\|f^n(\Psi-\Psi_\varepsilon)\|_{L^1([0,T]\times\mathbf{T}^3\times\mathbf{R}^3)}.$$

Since

$$\begin{aligned}&|\Psi(t,x,v)-\Psi_\varepsilon(t,x,v)|\\&\quad\le\|\Psi\|_{\infty}(1-\zeta_\varepsilon(t))+\int_{|z|\le 1}|\Psi(t,x,v)-\Psi(t,x,v-\varepsilon z)|\chi(z)dz,\end{aligned}$$

and $\sup_{t\ge 0}\|f^n(t)\|_{L^1_2}\le\mathcal{K}_0$, $0\le f^n\le 1$, it follows that

$$\begin{aligned}&\|f^n(\Psi-\Psi_\varepsilon)\|_{L^1([0,T]\times\mathbf{T}^3\times\mathbf{R}^3)}\\&\quad\le\|\Psi\|_{\infty}\mathcal{K}_0\int_0^T(1-\zeta_\varepsilon(t))dt+\Lambda_\Psi(\varepsilon)\le 4\|\Psi\|_{\infty}\mathcal{K}_0\varepsilon+\Lambda_\Psi(\varepsilon)\end{aligned} \tag{57}$$

where

$$\Lambda_\Psi(\varepsilon)=\sup_{|h|\le\varepsilon}\|\Psi-\Psi(\cdot,\cdot,\cdot+h)\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3\times\mathbf{R}^3)}.$$

Since $\Psi \in L^1(\mathbf{R}^1 \times \mathbf{T}^3 \times \mathbf{R}^3)$, this implies that $\Lambda_\Psi(\varepsilon) \to 0$ as $\varepsilon \to 0+$. On the other hand by writing $\chi_{\varepsilon,z}(v) = \chi_\varepsilon(v - z)$ we have

$$\langle f^n \Psi_\varepsilon \rangle(t, x) = \int_{\mathbf{R}^3} \langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(t, x)\Psi(t, x, z)dz$$

from which we deduce

$$\begin{aligned}
&|\langle f^n \Psi_\varepsilon \rangle(t + \tau, x + h) - \langle f^n \Psi_\varepsilon \rangle(t, x)| \\
&\quad \le \int_{\mathbf{R}^3} |\langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(t + \tau, x + h)||\Psi(t + \tau, x + h, z) - \Psi(t, x, z)|dz \\
&\quad + \int_{\mathbf{R}^3} |\langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(t + \tau, x + h) - \langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(t, x)||\Psi(t, x, z)|dz.
\end{aligned}$$

Using $0 \le f^n \le 1$ again gives $|\langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(t + \tau, x + h)| \le \|\chi_\varepsilon\|_{L^1(\mathbf{R}^3)} = 1$. This together with $|\Psi(t, x, z)| \le \|\Psi\|_\infty 1_{\{|z| \le R\}}$ imply that

$$\begin{aligned}
&\|\langle f^n \Psi_\varepsilon \rangle(\cdot + \tau,\ \cdot + h) - \langle f^n \Psi_\varepsilon \rangle\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3)} \\
&\quad \le \|\Psi(\cdot + \tau,\ \cdot + h, \cdot) - \Psi\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3 \times \mathbf{R}^3)} \\
&\qquad + \|\Psi\|_\infty \int_{|z| \le R} \|\langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(\cdot + \tau,\ \cdot + h) - \langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3)} dz.
\end{aligned}$$

By $\Psi \in L^1(\mathbf{R}^1 \times \mathbf{T}^3 \times \mathbf{R}^3)$ again, we have

$$\Lambda_\Psi(\tau, h) := \|\Psi(\cdot + \tau,\ \cdot + h, \cdot) - \Psi\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3 \times \mathbf{R}^3)} \to 0 \quad (|\tau| + |h| \to 0).$$

Also, since for any $z \in \mathbf{R}^3$ satisfying $|z| \le R$, the function $v \mapsto \chi_{\varepsilon,z}(v)$ belongs to $C_c^\infty(\mathbf{R}^3)$ with $\operatorname{supp} \chi_{\varepsilon,z} \subset \{v \in \mathbf{R}^3 \mid |v| \le \varepsilon + R\} \subset \{v \in \mathbf{R}^3 \mid |v| \le 2R\}$ and $\|\chi_{\varepsilon,z}\|_{1,\infty} = \|\chi_\varepsilon\|_{1,\infty}$, it follows from (42) that

$$\begin{aligned}
&\|\langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(\cdot + \tau,\ \cdot + h) - \langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3)} \\
&\quad \le C\|\zeta_\varepsilon\|_{1,\infty}\|\chi_\varepsilon\|_{1,\infty}(|\tau|^2 + |h|^2)^{\eta/2} \qquad \forall\, |z| \le R.
\end{aligned}$$

Here and below $C = (1 + A_0)^{1/2} C_{\mathcal{K}_0, 2R, T}$ denotes the constant as given in Theorem 1, i.e. $C_{\mathcal{K}_0, 2R, T}$ depends only on $\mathcal{K}_0$, R, T and on γ and $|\mathbf{T}^3|$. Thus

$$\begin{aligned}
&\int_{|z| \le R} \|\langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle(\cdot + \tau,\ \cdot + h) - \langle f^n \zeta_\varepsilon \chi_{\varepsilon,z} \rangle\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3)} dz \\
&\quad \le C\|\zeta_\varepsilon\|_{1,\infty}\|\chi_\varepsilon\|_{1,\infty}(|\tau|^2 + |h|^2)^{\eta/2}
\end{aligned}$$

and so we obtain for all $n \ge 1$,

$$\begin{aligned}
&\|\langle f^n \Psi_\varepsilon \rangle(\cdot + \tau,\ \cdot + h) - \langle f^n \Psi_\varepsilon \rangle\|_{L^1(\mathbf{R}^1 \times \mathbf{T}^3)} \\
&\quad \le \Lambda_\Psi(\tau, h) + C\|\Psi\|_\infty \|\zeta_\varepsilon\|_{1,\infty}\|\chi_\varepsilon\|_{1,\infty}(|\tau|^2 + |h|^2)^{\eta/2}. \qquad (58)
\end{aligned}$$

Summarizing (56)–(58) we get

$$\sup_{n\geq 1} \| \ |\langle f^n\Psi\rangle(\cdot+\tau,\ \cdot+h)-\langle f^n\Psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}$$

$$\leq 8\|\Psi\|_\infty\mathcal{K}_0\,\varepsilon+2\Lambda_\Psi(\varepsilon)+\Lambda_\Psi(\tau,h)+C\|\Psi\|_\infty\|\zeta_\varepsilon\|_{1,\infty}\|\chi_\varepsilon\|_{1,\infty}(|\tau|^2+|h|^2)^{\eta/2}.$$

This gives

$$\limsup_{|\tau|+|h|\to 0}\ \sup_{n\geq 1}\|\langle f^n\Psi\rangle(\cdot+\tau,\ \cdot+h)-\langle f^n\Psi\rangle\|_{L^1(\mathbf{R}^1\times\mathbf{T}^3)}\leq 8\|\Psi\|_\infty\mathcal{K}_0\,\varepsilon+2\Lambda_\Psi(\varepsilon).$$

Letting $\varepsilon\to 0$ leads to (55). This completes the proof. □

ACKNOWLEDGMENTS

I am grateful to a referee who let me know an earlier work of Alexandre[(2)] on the Boltzmann–Fermi–Dirac model and gave a comment on the full compactness of weak solutions. This work was supported by National Natural Science Foundation of China, Grant No. 10171053.

REFERENCES

1. R. A. Adams, *Sobolev Spaces* (Academic Press, New York, 1975).
2. R. Alexandre, On some related non homogeneous 3D Boltzmann models in the non cutoff case, *J. Math. Kyoto Univ.* **40**(3):493–524 (2000).
3. R. Alexandre, L. Desvillettes, C. Villani and B. Wennberg, Entropy dissipation and long-range interactions, *Arch. Rational Mech. Anal.* **152**:327–355 (2000).
4. R. Alexandre and C. Villani, On the Boltzmann equation for long-range interactions, *Commun. Pure Appl. Math.* **55**(1):30–70 (2002).
5. L. Arkeryd, Intermolecular forces of infinite range and the Boltzmann equation, *Arch. Ration. Mech. Anal.* **77**:11–21 (1981).
6. D. Benedetto, F. Castella, R. Esposito and M. Pulvirenti, Some considerations on the derivation of the nonlinear quantum Boltzmann equation, *J. Stat. Phys*. **116**:381–410 (2004).
7. D. Bohm, *Quantum Theory* (Prentice-Hall, New York, 1951).
8. F. Bouchut and L. Desvillettes, Averaging lemmas without time Fourier transform and application to discretized kinetic equations. *Proc. R. Soc. Edinburgh Sect. A* **129**:19–36 (1999).
9. F. Bouchut, F. Golse, and M. Pulvienti, *Kinetic Equations and Asymptotic Theory*, B. Perthame and L. Desvillettes, eds., Series in Applied Mathematics 4 (Gauthier-Villar, Paris, 2000).
10. F. Bouchut, Hypoelliptic regularity in kinetic equations, *J. Math. Pure. Appl.* **81**(9):1135–1159 (2002).
11. T. Carleman, *Problèmes Mathématiques dans la Théorie Cinétique des Gaz* (Almqvist and Wiksell, Uppsala, 1957).
12. E. Carlen and W. Gangbo, Solution of a model Boltzmann equation via steepest descent in the 2-Wasserstein metric, *Arch. Ration. Mech. Anal.* **172**:21–64 (2004).
13. C. Cercignani, *The Boltzmann Equation and Its Applications* (Springer-Verlag, New York, 1988).
14. C. Cercignani, R. Illner, and M. Pulvirenti, *The Mathematical Theory of Dilute Gases* (Springer-Verlag, New York, 1994).

15. S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 3rd edn. (Cambridge University Press, Cambridge, UK, 1970).
16. R. J. DiPerna and P. L. Lions, On the Cauchy problem for the Boltzmann equation: Global existence and weak stability, *Ann. Math.* **130**:321–366 (1989).
17. R. J. DiPerna, P. L. Lions, Y. Meyer, L^p regularity of velocity averages, *Ann. Inst. H. Poincarf Anal. Non Linaire*, **8**:271–287 (1991).
18. J. Dolbeault, Kinetic models and quantum effects: a modified Boltzmann equation for Fermi–Dirac particles, *Arch. Ration. Mech. Anal.* **127**:101–131 (1994).
19. F. Golse, P. L. Lions, B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation. *J. Funct. Anal.* **76**:110–125 (1988).
20. T. Goudon, On Boltzmann equations and Fokker-Planck asymptotics: influence of grazing collisions, *J. Stat. Phys.* **89**:751–776 (1997).
21. P. L. Lions, Compactness in Boltzmann's equation via Fourier integral operators and applications, III, *J. Math. Kyoto Univ.* **34**(3):539–584 (1994).
22. X. G. Lu, On spatially homogeneous solutions of a modified Boltzmann equation for Fermi–Dirac particles, *J. Statist. Phys.* **105**:353–388 (2001).
23. X. G. Lu and B. Wennberg, On stability and strong convergence for the spatially homogeneous Boltzmann equation for Fermi–Dirac particles, *Arch. Ration. Mech. Anal.* **168**:1–34 (2003).
24. X. G. Lu, On the Boltzmann Equation for Fermi–Dirac Particles with Very Soft Potentials: Global Existence of Weak Solutions, manuscript in preparation.
25. L. W. Nordheim, On the kinetic methods in the new statistics and its applications in the electron theory of conductivity, *Proc. R. Soc. London Ser. A* **119**:689 (1928).
26. E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton University Press, Princeton, NJ, 1971).
27. E. A. Uehling and G. E. Uhlenbeck, Transport phenomena in Einstein-Bose and Fermi–Dirac gases, I, *Phys. Rev.* **43**:552–561 (1993).
28. C. Villani, On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations, *Arch. Ration. Mech. Anal.* **143**:273–307 (1998).
29. C. Villani, *Handbook of Mathematical Fluid Dynamics*, Vol. **I**:71–305, North-Holland, Amsterdam, 2002.
30. B. Wennberg, Regularity in the Boltzmann equation and the Radon transform, *Comm. Partial Differential Equations* **19**:2057–2074 (1994).